Now thm A is a corollary of thmB and the following two propositions:

Thm 3.6.2 Any regular and $\overset{fs}{\text{log}}$ smooth scheme over S is punctually pure.

Proposition 3.5.7 X, Y: S-scheme of finite type, integral regular and admits ample invertible sheaves. Let $X \xrightarrow{f} Y$ be a proper, dominant and generically étale morphism of degree d ~~pub~~ invertible in $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. Then the punctually purity ~~for~~ of X implies that of Y.

$\boxed{\text{proof of proposition 3.5.7}}$

$\boxed{\text{Step 1}}$ Any smooth S-scheme is punctually pure

S punctually pure $\Rightarrow (S, s)$ pure $\overset{\text{smooth base change thm}}{\Rightarrow} (\mathbb{A}^n_S, \mathbb{A}^n_s)$ pure   But $\mathbb{A}_{\mathbb{Z}}$ is punctually pure $\Rightarrow \mathbb{A}^n_S$ is punctually pure.

$\Rightarrow$ any smooth S-scheme is p.pure

$\boxed{\text{Step 2}}$ X is punctually pure if $X \xrightarrow{p} S$ the Gysin map $cl_{X/S}: \Lambda_X \longrightarrow K_X := p^{!}\Lambda_S$ in $D^+(X_{et}, \Lambda)$ is an isomorphism

Indeed, $X \overset{i}{\hookrightarrow} Z$, $p\downarrow$, $\swarrow q$: smooth. S

thm 2.5.12

$cl_p = i^{!}(cl_q) \circ cl_i$ is an isom iff cl_i is an isom. iff (X, Z) is pure

Since cl_q is an isom

By prop 3.2.3 and that Z is punctually pure iff X is punctually pure.

$\boxed{\text{Step 3}}$ Consider the following comm. diag in $D(X_{et}, \Lambda)$

$$\begin{array}{ccccc}
\Lambda_Y & \longrightarrow & Rf_*\Lambda_X & \longrightarrow & \Lambda_Y \\
{\scriptstyle cl_{Y/S}}\downarrow & & \downarrow{\scriptstyle Rf_*(cl_{X/S})} & & \downarrow{\scriptstyle cl_{Y/S}} \\
K_Y & \longrightarrow & Rf_*\Lambda_X & \longrightarrow & K_Y
\end{array}
\quad \underset{f:\text{proper}}{\overset{\text{adj}}{\longleftrightarrow}} \quad
\begin{array}{ccccc}
f^*\Lambda_Y & \xrightarrow{\sim} & \Lambda_X & \xrightarrow{cl_f} & f^{!}\Lambda_Y \\
{\scriptstyle f^*cl_{Y/S}}\downarrow & & \downarrow{\scriptstyle cl_{X/S}} & & \downarrow{\scriptstyle f^{!}cl_{Y/S}} \\
f^*K_Y & \longrightarrow & K_X & \xrightarrow{\sim} & f^{!}K_Y
\end{array}$$

By Lemma below, $cl_{Y/S}$ is a direct factor of $Rf_*(cl_{X/S})$. Thus if $cl_{X/S}$ is an isom, then $cl_{Y/S}$ is also an isom. $\boxtimes$

$\boxed{\text{Lemma 3.5.9}}$ The composition $\Lambda_Y \to Rf_*\Lambda_X \to \Lambda_Y$ and $K_Y \longrightarrow Rf_*K_X \longrightarrow K_Y$ are given by multiplication by d.

proof Y connected $\Rightarrow \Lambda_Y \cong \mathrm{End}_{D^+(Y_{et}, \Lambda)}(\Lambda_Y, \Lambda_Y)$

Locally biduality thm $G \cong DDG \Rightarrow \Lambda_Y \cong DD\Lambda_Y \cong R\mathcal{H}om(K_Y, K_Y)$ or $\Lambda_Y \cong \mathrm{End}_{D^+(Y_{et}, \Lambda)}(K_Y)$.

Thus it suffices to prove ~~the Lem~~ the conclusion over an non-empty open subset of Y.

$\Rightarrow$ WMA f is a finite étale covering $\Rightarrow$ WMA $X = \coprod_{d\text{-copies}} Y$.

Now we only need to prove theorem 3.6.2. We first introduce Log Sch and Log smooth, then we show that fs-log smooth $X \longrightarrow S$

étale locally on   étale

§1 Log scheme

M : monoid (commutative with unit)

M^{gp} = associated group of M = $\{ab^{-1} \mid a,b \in M\}/\sim$ ⟵ $ab^{-1} = cd^{-1}$ iff $sad = sbc$ for some $s \in M$.

Def A log scheme X is a scheme X with a log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$. We usually write $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{O}_X^*$.

morphism of sheaves of monoids

$\mathcal{O}_X$: multiplication.

Any map $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ (prelog structure), we can associated a log structure $\alpha^a : \mathcal{M}_X^a \to \mathcal{O}_X$ by the following push out in the cat. of sheaves of monoids on X_{et}.

$$\begin{tikzcd}
\alpha^{-1}(\mathcal{O}_X^*) \arrow[r, hook] \arrow[d] & \mathcal{M}_X \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
\mathcal{O}_X^* \arrow[r, hook] \arrow[drr, bend right] & \mathcal{M}_X^a \arrow[dr, "\alpha^a"] & \\
& & \mathcal{O}_X
\end{tikzcd}$$

Note that $\mathcal{M}_X^a/\mathcal{O}_X^* \cong \mathcal{M}_X/\alpha^{-1}(\mathcal{O}_X^*)$

$LogSch$ = Cat. of log schemes with morphism $f : X \to Y$ given by a morphism of schemes $f : X \to Y$ such that

$$\begin{tikzcd}
f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}_X \arrow[d] \\
f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X
\end{tikzcd}$$

Definition (pull-back of log structure) $f : X \to Y$.

$f^*\mathcal{M}_Y$ = pull-back of $\mathcal{M}_Y$ = log. str. associated to the prelog structure $f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Note that $f^*\mathcal{M}_Y/\mathcal{O}_X^* \cong f^{-1}(\mathcal{M}_Y/\mathcal{O}_Y^*)$ or $\overline{f^*\mathcal{M}_Y} = f^{-1}\overline{\mathcal{M}_Y}$.

Example of log schemes 1) trivial log ~~Stures~~ structure $\mathcal{M}_X = \mathcal{O}_X^* \subseteq \mathcal{O}_X$. ⇝ $Sch \to LogSch$

2) X regular, D : reduced normal crossing divisor, $\mathcal{M}_X = \mathcal{O}_X \cap j_*\mathcal{O}_U$, $j : U \hookrightarrow X$, $U = X \setminus D$.

3). (toric) For a monoid P, let $S[P] = \operatorname{Spec} \mathbb{Z}[P]$ denote the log scheme with the log structure associated to $P \to \mathbb{Z}[P]$.

Then for any log scheme X,

$$\operatorname{Hom}_{logSch}(X, S[P]) = \operatorname{Hom}(P, \Gamma(X, \mathcal{M}_X))$$

Any strict map $X \to S[P]$ is called a chart for X ...

Definition (fine Log / fs log)

1) A monoid M is called integral if $M \longrightarrow M^{gp}$ is injective

(finitely generated if $\exists r \in \mathbb{N}$ such that $\mathbb{N}^r \twoheadrightarrow M$)

fine = integral + finitely generated.

2) A monoid M is called <u>saturated</u> if M is integral and M equals to the saturation

$$M^{sat} = \{ m \in M^{gp} \mid \exists n \geq 1 \text{ such that } m^n \in M \}$$

fs = fine + saturated.

3) A log scheme is called fine (or fs) if étale locally on X, there is a chart $X \longrightarrow S[P]$ such that P is fine (or fs).

Remark $X \longrightarrow Y$ fs log, étale locally on X and Y

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ S[P] \arrow[r] & S[Q] \end{tikzcd}$$

<u>Definition</u> (Olsson) Let $\mathbb{P}$ be a property of morphisms of schemes which is étale local on the source

e.g. $\mathbb{P}$ = smooth, étale, flat, ...

Let $X \xrightarrow{f} S$ be a morphism between fine log schemes.

$\mathcal{L}og_S$ = cat. of fine log schemes over S with "strict map" as morphisms.

Then $\mathcal{L}og_S$ is an alg. stack which is locally of finite presentation over S.

We say f is log $\mathbb{P}$ iff $\mathcal{L}og_X \longrightarrow \mathcal{L}og_S$ is $\mathbb{P}$

Remark If $X \xrightarrow{f} Y$ is strict, then f is log $\mathbb{P}$ iff f is $\mathbb{P}$.

重要

Definition of log smooth by deformation

A morphism $X \xrightarrow{f} Y$ of fine log schemes is called <u>log smooth</u> if the underlying morphism $X \longrightarrow Y$ is locally of finite presentation and if for any commutative diagram of fine log schemes:

$$\begin{tikzcd} T' \arrow[r, "s"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "t"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

where i is an <u>exact</u> closed immersion which is defined by an ideal sheaf $\mathcal{I}$ of T such that $\mathcal{I}^2 = (0)$.

<u>Then</u> étale locally on T, there is a map $g: T \longrightarrow X$ such that $g \circ i = s$ and $f \circ g = t$.

standard example of smooth morphism $\mathbb{A}^n_S \to S$

(6)

Proposition (Standard example of log smooth)

Let P and Q be fine monoids with homomorphism $Q \xrightarrow{f} P$.

R: ring.

Assume that ~~(Q)~~ $\ker(f^{gp}: Q^{gp} \to P^{gp})$ and $\left(\mathrm{Coker}(Q^{gp} \to P^{gp})\right)_{tor}$ are finite groups of orders invertible ~~in~~ R.

Then $X = \mathrm{Spec}\, R[P] \longrightarrow Y = \mathrm{Spec}\, R[Q]$ is log smooth with canonical log structure.

proof Consider

$$\begin{tikzcd} T' \arrow[r, "s"] \arrow[d, "i"] & X = \mathrm{Spec}\, R[P] \arrow[d, "f"] \\ T \arrow[r, "t"] & Y = \mathrm{Spec}\, R[Q] \end{tikzcd}$$

with $T' \xrightarrow{i} T$ exact closed immersion, defined by I ~~=~~ with $I^2 = 0$.

Then we have

due to $I^2 = 0$

since exact $\Rightarrow \overline{\mathcal{M}}_T \cong \overline{\mathcal{M}}_{T'}$

$$\begin{tikzcd} 1 \arrow[r] & 1+I \arrow[r] \arrow[d, equal] & \mathcal{M}_T \arrow[r] \arrow[d, hook] & \mathcal{M}_{T'} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & 1+I \arrow[r] & \mathcal{M}_T^{gp} \arrow[r] & \mathcal{M}_{T'}^{gp} \arrow[r] & 1 \\ 1 \arrow[r] & K \arrow[r] \arrow[u, "u"] & Q^{gp} \arrow[r, "f^{gp}"] \arrow[u, "v"] & P^{gp} \arrow[r] \arrow[u, "w"] & C \arrow[r] & 1 \end{tikzcd}$$

$u(x) = 1+a$

$x \mapsto$ $1+x$, I

$0 = x^l \longmapsto u(x)^l = 1+la = 1$

$\Rightarrow l \cdot a = 0 \Rightarrow a = 0$. kernel

$\mathrm{Im}(f^{gp})$, a'

cokernel

$R[P] \longrightarrow \mathcal{O}_T$

P

$\boxed{\text{We show}}$ etale locally on T, $\exists$ map $P^{gp} \to \mathcal{M}_T^{gp}$ (then it induce the map $P \to \mathcal{M}_T$ which induces the desired morphism $T \longrightarrow X = \mathrm{Spec}\, R[P]$)

$\boxed{\text{Since the order of } K \text{ is invertible in } R \Rightarrow u = 1}$

$\Rightarrow \exists a' : \mathrm{Im} f^{gp} \longrightarrow \mathcal{M}_T^{gp}$ making the diagram commutes

$$\begin{tikzcd} 1 \arrow[r] & 1+I \arrow[r] & \mathcal{M}_T^{gp} \arrow[r] & \mathcal{M}_{T'}^{gp} \arrow[r] & 1 \\ 1 \arrow[r] & \mathrm{Im} f^{gp} \arrow[r] \arrow[ur, "a'"] & P^{gp} \arrow[r] \arrow[u] \arrow[ul, dashed, "a''"] & C \arrow[r] & 1 \end{tikzcd}$$

$1+I \overset{I}{\underset{\text{is}}{}}$

$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p, B) = {}_pB$

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, B) = B/pB$

~~We~~ show there exists a morphism $a'' : \mathrm{Im} f^{gp} \to \mathcal{M}_T^{gp}$ with commutative condition (obvious)

$\mathbb{Z}$, proj $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, B) =$

The obstruction for the existence of a'' lies in $\mathrm{Ext}^1_{\mathbb{Z}}(C, \overset{1+I}{I})$.

In general, if a positive integer n is invertible in R (I), then $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, I) = 0$

Combining this with $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, I) = 0$ we have $\mathrm{Ext}^1(C, I) = 0$. Thus $\underline{a'' : P^{gp} \to \mathcal{M}_T^{gp} \text{ exists}}$

$\mathrm{Hom}(\mathrm{Im} f^{gp}, \mathcal{M}_T^{gp}) \ni a' \longrightarrow \mathrm{Hom}(\mathrm{Im} f^{gp}, \mathcal{M}_{T'}^{gp}) \longrightarrow \mathrm{Ext}(\mathrm{Im} f^{gp}, I)$

Theorem B(~~Toroidal~~ Toroidal characterization of log smooth morphism)

Let $f: X \longrightarrow Y$ be a morphism in $LSch^f$ and with a chart $Y \to S[Q]$
$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\text{strict}"] \\ & S[Q] \end{tikzcd}$$
$LSch^{fs}$

Q : fine monoid. fs monoid

then f is log smooth if and only if étale locally on X, there exist

$\Leftarrow$ trivial

$$\begin{tikzcd} X \arrow[r, "\text{strict étale}"] \arrow[dr] & Y \times_{S[Q]} S[P] \arrow[r, "\text{strict}"] \arrow[d] \arrow[dr, phantom, "\square"] & S[P] \arrow[d] \\ & Y \arrow[r, "\text{strict}"] & S[Q] \end{tikzcd}$$

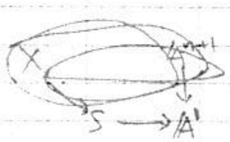
P : fine monoid
fs monoid
We can require $Q^{gp} \to P^{gp}$ to be injective

where $S[P] \longrightarrow S[Q]$ is a standard log smooth map between fine monoid.

比较

$X \xrightarrow{\text{smooth}} S$ if Zariski locally on X

$$\begin{tikzcd} X \arrow[r, "\text{étale}"] \arrow[d] \arrow[dr] & \mathbb{A}^n_S \arrow[r] \arrow[d] & \mathbb{A}^n_{\mathbb{Z}} \arrow[d] \\ \, & S \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$



In $LSch^{fs}$

[Prop 3.6.3] Moreover, if Q is a fs-monoid torsion free, then we can choose P to be a torsion free fs monoid.

$\operatorname{Spec} R[P] \longrightarrow \operatorname{Spec} R[Q]$ is log smooth
where $R = \mathbb{Z}[\frac{1}{N_1 N_2}]$ $N_1 = \# \ker f^{gp}$
$N_2 = \# \operatorname{tor}(\operatorname{coker} f^{gp})$

[Prop 3.6.5] Let S be a trait with canonical log structure $S \longrightarrow S[\mathbb{N}]$ defined by a uniformizer π of $\mathcal{O}_S$. Let $(X, \mathcal{M}_X) \xrightarrow{f} (S, \mathcal{M}_S)$ be a log smooth morphism in $LSch^{fs}$.

If X is regular, then locally for the étale topology on X, X admits an étale covering to a scheme $V(S, \pi, e_1, \ldots, e_n) = \operatorname{Spec} \dfrac{\mathcal{O}_S[T_1, \ldots, T_n]}{(T_1^{e_1} \cdots T_n^{e_n} - \pi)}$

with $(e_1, \ldots, e_n) \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$

[Remark] log regular + regular $\Rightarrow$ $\forall \bar{x} \to X$, $\overline{\mathcal{M}}_{X,\bar{x}} \cong \mathbb{N}^r$ for some r.

$\mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X$ pre log
$\mathcal{M}/\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{M}^a/\mathcal{O}_X^*$
$f^*\mathcal{M}_Y/\mathcal{O}_X^* \cong f^{-1}(\mathcal{M}_Y \ldots$

proof Theorem B $\exists$ P: torsion free fs monoid

$$\begin{tikzcd} X \arrow[r, "\text{étale}"] \arrow[dr] & S \times_{S[\mathbb{N}]} S[P] \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[P] = S[P] \arrow[d] \\ & S \arrow[r] & \operatorname{Spec}\mathbb{Z}[\mathbb{N}] = S[\mathbb{N}] \end{tikzcd}$$

doesnot change log structure

after replace P by $P[P']$
WMA P' is a group

Fix $\bar{x} \to X$ geo. point. Let $P' = \left\{ p \in P \,\middle|\, \begin{array}{l} \text{Image of } p \text{ in } \Gamma(X, \mathcal{O}_X) \\ \text{is invertible at } \bar{x} \end{array} \right\}$

$\overset{\text{étale locally}}{=} \{ p \in P \mid p \text{ is invertible in } \mathcal{O}_X \}$

$X \longrightarrow \operatorname{Spec}\mathbb{Z}[P]$ is a chart $\Rightarrow$ $P' \subseteq \ker(P^{gp} \to \overline{\mathcal{M}}^{gp}_{X,\bar{x}})$ and $P/P' \cong \overline{\mathcal{M}}_{X,\bar{x}} \overset{\text{for some } r}{\cong} \mathbb{N}^r$

Then $P \cong \mathbb{N}^r \oplus P'$ and

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec}\mathbb{Z}[P] = (\text{affine space}) \times (\text{torus})^r \\ \downarrow & & \downarrow \\ S & \longrightarrow & \operatorname{Spec}\mathbb{Z}[\mathbb{N}] \cancel{\cong \mathbb{A}} = \operatorname{Spec}\mathbb{Z}[T] \end{array}$$

$$\mathbb{N} \longrightarrow P \cong \mathbb{N}^r \oplus P' \cong \mathbb{N}^r \oplus \mathbb{Z}^s$$

$$1 \longmapsto (e_1, \dots, e_r, f_1, \dots, f_s)$$ Choose basis $a_1, \dots, a_s$ of P' such that $f_i \in \mathbb{N}$

Then $X \xrightarrow{\text{étale}} V(S, \pi, e_1, \dots, e_r, f_1, \dots, f_s)$ with $e_1, \dots, e_r, f_1, \dots, f_s$ not all zero

注意 $S \times_{\operatorname{Spec}\mathbb{Z}[\mathbb{N}]} \operatorname{Spec}\mathbb{Z}[P] = \operatorname{Spec} \dfrac{\mathcal{O}_S[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_{r+s}^{\pm 1}]}{(T_1^{e_1} \cdots T_r^{e_r} T_{r+1}^{f_1} \cdots T_{r+s}^{f_s} - \pi)}$

$X \xrightarrow{\text{étale}}$

$\Big\downarrow$ open

$\operatorname{Spec} \dfrac{\mathcal{O}_S[T_1, \dots, T_r, T_{r+1}, \dots, T_{r+s}]}{(T_1^{e_1} \cdots T_r^{e_r} T_{r+1}^{f_1} \cdots T_{r+s}^{f_s} - \pi)}$

$\boxed{\text{sketch of the proof of thm B}}$ $\omega^1_{X/Y} = \Omega^1_{(X, M_X)/(Y, M_Y)} = \dfrac{\Omega^1_{X/Y} \oplus \mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp}/f^* M_Y^{gp}}{\langle d(\alpha(m)), -\alpha(m) \otimes m \rangle_{m \in M_X}}$ $\alpha: M_X \to \mathcal{O}_X$ is locally free.

$\mathcal{O}_X \otimes_{\mathbb{Z}} M_X \xrightarrow[\text{dlog}]{} \omega^1_{X/Y}$ surj. $\quad \operatorname{dlog} a = \text{class of } (0, 1 \otimes a) \in \omega^1_{X/Y}$.

Take $t_1, \dots, t_r \in M_{X, \bar{x}}$, such that $\{\operatorname{dlog} t_i\}_{i=1}^r$ is a basis of $\omega^1_{X/Y, \bar{x}}$ over $\mathcal{O}_{X, \bar{x}}$.

$H := \mathbb{N}^r \oplus Q \xrightarrow{\varphi} M_{X, \bar{x}}$ with $Q \to (f^{-1} M_Y)_{\bar{x}} \to M_{X, \bar{x}}$ and $\mathbb{N}^r \longrightarrow M_{X, \bar{x}}$, $(n_1, \dots, n_r) \longmapsto t_1^{n_1} \cdots t_r^{n_r}$.

$k(\bar{x}) \otimes_{\mathbb{Z}} \mathbb{Z}^r \longrightarrow k(\bar{x}) \otimes_{\mathbb{Z}} \overline{M}_{X, \bar{x}}^{gp} / f^{-1} \overline{M}_{Y, \bar{x}}^{gp}$ is surjective.

$\Rightarrow k(\bar{x}) \otimes_{\mathbb{Z}} H^{gp} \longrightarrow k(\bar{x}) \otimes_{\mathbb{Z}} \overline{M}_{X, \bar{x}}^{gp}$ is surjective.

$C := \operatorname{Coker}(\varphi^{gp})$ is annihilated by an integer N invertible in $\mathcal{O}_{X, \bar{x}}$. $\quad \mathcal{O}_{X, \bar{x}}$ is N-divisible

Take $a_1, \dots, a_d \in M_{X, \bar{x}}^{gp}$ which generate C. Then we may assume $a_i^N = \varphi(b_i)$

$$\begin{array}{ccccc} & e_i & \longmapsto & N e_i & \\ e_i & \mathbb{Z}^d & \longrightarrow & \mathbb{Z}^d & \\ \downarrow & \downarrow & & \downarrow & \\ b_i & H^{gp} & \longrightarrow & G & \end{array}$$

$\boxed{\phi: G \to M_{X, \bar{x}}^{gp} \text{ which map } G \text{ surj to } \overline{M}_X}$

Take $P = \phi^{-1}(M_{X, \bar{x}})$

$\operatorname{Coker}(H^{gp} \to M_{X, \bar{x}}^{gp})$ torsion free

Theorem 3.6.2 is a corollary of the following two propositions:

$\boxed{\text{prop 3.6.5}}$ Let X be a fs-log scheme over S. If X is regular and log smooth over S, then étale locally on X, X admits an étale covering to a scheme

$$V(S,\pi,e_1,\dots,e_n) = \mathrm{Spec}\,\frac{\mathcal{O}_S[T_1,\dots,T_n]}{(T_1^{e_1}\cdots T_n^{e_n}-\pi)} \quad \text{with } (e_1,\dots,e_n)\in \mathbb{N}^n \setminus \{(0,\dots,0)\}$$

$\boxed{\text{prop 3.5.12}}$ As above, $V(S,\pi,e_1,\dots,e_n)$ is regular and punctually pure.

$\boxed{\text{Remark}}$ If X/S is only log smooth, then by Kato-Niziol's resolution of singularities: there is a log blow-up $X'\longrightarrow X$ such that X' is regular and log smooth over S

in particular, $X'\longrightarrow X$ is proper, birational, log étale, generically étale .... (~~Purity for X' implies the purity for~~

$\boxed{\text{proof of proposition 3.5.12}}$ —— WMA: $e_i\neq 0$ for all i (cf. Corollary 3.5.6)

—— WMA $(e_1,\dots,e_n)=1$.

Indeed, put $d=(e_1,\dots,e_n)$. $S'=\mathrm{Spec}\,\frac{\mathcal{O}_S[T]}{(T^d-\pi)}$ with $\pi'=T$. Then

$$V(S,\pi,e_1,\dots,e_n) = V(S',\pi',\tfrac{e_1}{d},\dots,\tfrac{e_n}{d}).$$

—— Since $\prod_{i=1}^n T_i^{e_i}-\pi\notin m^2$ for all local ring m of $V(S,\pi,e_1,\dots,e_n)$ at some closed points, thus V is regular with smooth generic fiber.

—— Let $H_i=V(T_i)=$"$T_i=0$". Then H_i is an affine space which is punctually pure.

Then we only need to show that (V,H_i) is pure for each $i=1,\dots,n$.

—— We may assume $\ell\nmid e \overset{\text{def}}{=}$ least common multiple of $(e_i)_{i=1}^n$.

Indeed, if $\ell\mid e_1$ for example, then $V:=V(S,\pi,e_1,\dots,e_n)\xrightarrow[\text{flat}]{\text{finite}} V(S,\pi,\frac{e_1}{\ell},\dots,e_n)=:V'$

$V\to V'$ is étale outside H_1 of degree ℓ.

By prop 3.4.5 $\Rightarrow (V,H_1)$ is pure iff (V',H_1) is pure.

—— Now $\ell\nmid e$, $V(S,\pi,e,\dots,e)\xrightarrow[\text{flat}]{\text{finite}} V(S,\pi,e_1,\dots,e_n)$ is of degree $\frac{e^n}{e_1\cdots e_n}$ which is prime to ℓ. $T_i^{e/e_i} \longleftarrow\!\shortmid T_i$

By prop 3.4.13 and prop 3.5.7 $\Rightarrow (V,H_i)$ is pure if and only if $(V(S,\pi,e,\dots,e),H_i)$ is pure

—— $V(S,\pi,e,\dots,e)=V(S',\pi',1,\dots,1)=\mathrm{Spec}\,\frac{\mathcal{O}_{S'}[T_1,\dots,T_n]}{(T_1\cdots T_n-\pi')}$ is punctually pure.

where $S'=\frac{\mathcal{O}_S[T]}{(T^e-\pi)}$, $\pi'=T$.