(X,D) log smooth, mean that X/F smooth, $D \subseteq X$ SNC divisor. $\rightsquigarrow$ We also write (X,U,D), $U = X\setminus D \overset{j}{\hookrightarrow} X$

(of characteristic $p>0$, $\ell \neq p$.)

Notation F: perfect field. All schemes/F are assume to be separated of finite type/F

§0 Recall CC and Swan class

(X,U,D) log smooth, $\mathcal{F}$: smooth sheaf on $U \overset{j}{\hookrightarrow} X$, characteristic cycle $CC(j_!\mathcal{F}) \in CH_d(X)$, $d = \dim(X)$.

$$Sw^{cc}(\mathcal{F}) = \langle CC(j_!\mathcal{F}) - \operatorname{rank}\mathcal{F}\, CC(j_!\mathbb{F}_\ell), X\rangle \in CH_0(D)$$

GOS formula. $\chi_c(X,\mathcal{F}) = \operatorname{rank}\mathcal{F}\cdot \chi_c(X) - \deg Sw^{cc}(\mathcal{F})$.

Today we construct $Sw(\mathcal{F}) \in CH_0(X\setminus U)\otimes\mathbb{Q}$ using another method, Conjecture: $Sw(\mathcal{F}) = Sw^{cc}(\mathcal{F})$.

§1. Classical case of GOS formula

Definition 1.1 (Classical definition of Swan conductor)

K: complete discrete valuation ring with perfect residue field F.

L/K finite separable extension with residue field extension E/F, let $G = Gal(L/K)$

Assume that the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L is a discrete valuation ring.

Swan character $G \xrightarrow{Sw} \mathbb{Z}$

$$\sigma \longmapsto Sw_{L/K}(\sigma) = \begin{cases} \operatorname{length}_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} - ([L:K] - [E:F]) & \text{if } \sigma = 1 \\ -\operatorname{length}_{\mathcal{O}_K} \dfrac{\mathcal{O}_L}{\left(\frac{\sigma(x)}{x} - 1 \,;\, x \in L^\times\right)} & \text{if } \sigma \neq 1. \end{cases}$$

$\boxed{\sum_{\sigma\in G} Sw_{L/K}(\sigma) = 0}$

V: ℓ-adic rep. of G, put $Sw(V) \overset{\text{def}}{=} \frac{1}{[L:K]} \sum_{\sigma \in P} Sw_{L/K}(\sigma)\, Tr(\sigma : V) \in \mathbb{Z}$ (Swan conductor)

$P \leftarrow$ p-sylow subgroup (or elements of order pth power

Remark 1) π: prime element of L, $\sigma \in P$

the ideal $\left(\frac{\sigma(x)}{x} - 1 \,;\, x\in L^\times\right)$ is generated by $\frac{\sigma(\pi)}{\pi} - 1$. $\Rightarrow Sw(\sigma) = -\operatorname{ord}_L\left(\frac{\sigma(\pi)}{\pi} - 1\right)$ for $\sigma \neq 1$.

2) $-Sw_{L/K}(\sigma) > 0 \Leftrightarrow \sigma \in P_{L/K}\setminus\{1\}$

Definition 1.2 (Global formulation of Swan class on Curves)

(X,U,D) ~~smooth~~ log smooth curve, X: proper

$\mathcal{F}$: smooth sheaf of $\mathbb{F}_\ell$-modules on U.

Assume that $\mathcal{F}$ is trivialized by a finite Galois covering $V \xrightarrow{f} U$ of Galois group G,

Let Y be the normalization of X in V.

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \downarrow & \square & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Let $(Y\times Y)' \longrightarrow Y\times Y$ be the log blow-up at (y,y) for each $y \in Y\setminus V$.

For $\sigma \in G\setminus\{1\}$, let Γ_σ be the graph of $\sigma \subseteq V\times_U V$, $\overline{\Gamma_\sigma}$ be the closure of Γ_σ in $(Y\times Y)'$.

$$G \longrightarrow CH_0(Y\setminus V)$$

$$\sigma \longmapsto Sw_{V/U}(\sigma) = \begin{cases} -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(Y\setminus V) & \text{if } \sigma \neq 1 \\ -\sum_{\sigma\neq 1} Sw_{V/U}(\sigma) & \end{cases}$$

Thm 1.3 (Hasse-Arf) $\mathrm{Sw}(\mathcal{F}) \in CH_0(X \setminus U)$

Thm 1.4 (GOS) $\chi_c(U, \mathcal{F}) = \chi_c(U)\,\mathrm{rank}\,\mathcal{F} - \deg \mathrm{Sw}(\mathcal{F})$

Remark 1.5 $(\Gamma_\sigma, \Delta_Y)_{(Y\times Y)'} = \sum_{\substack{y \in Y \setminus V \\ \sigma(y)=y}} \mathrm{length}_{\widehat{\mathcal{O}_{Y,y}}} \dfrac{\widehat{\mathcal{O}_{Y,y}}}{\left(\frac{\sigma(b)}{b} - 1 ; b \in \widehat{\mathcal{O}_{Y,y}} \setminus \{0\}\right)} \cdot [y]$

$\mathrm{Tr}(\sigma^* : H_c^*(V_{\bar F}, \mathbb{Q}_\ell))$ — Trace formula for open variety.

proof of fact will be given latter.

§2 log. blow-up and log. product

We don't use log geometry in our talk. (in relative version $X/\mathcal{O}_K$, then you really needs log geometry)

Definition 2.1 X/F with a finite family of Cartier divisors $D = (D_i)_{i\in I}$

Let $(X\times X)'_i \longrightarrow X\times X$ be the blow-up at $D_i \times D_i$

$$(X\times X)'_i \hookleftarrow (X\times X)^\sim_i = (X\times X)'_i \setminus \left\{\begin{array}{l}\text{proper transforms of} \\ D_i\times X \text{ and } X\times D\end{array}\right\}$$

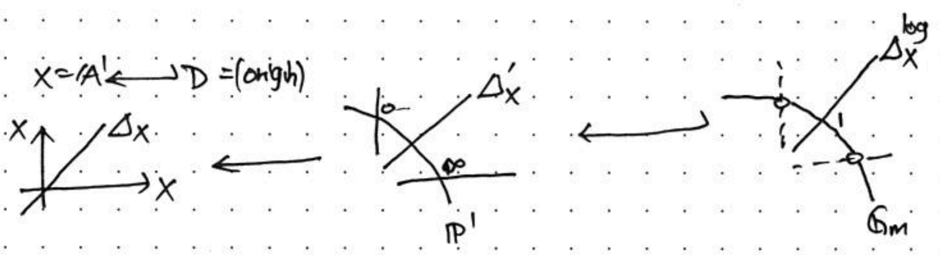
log blow up $(X\times X)' \overset{\text{def}}{=\!=} \prod_{i\ X\times X} (X\times X)'_i \xleftarrow{\Delta'_X} X$

$$\begin{array}{ccc} (X\times X)' & \xleftarrow{\Delta'_X} & X \\ \downarrow & \swarrow_{\Delta_X} & \\ X\times X & & \end{array}$$

log product $(X\times X)^\sim \overset{\text{def}}{=\!=} \prod_{i\ X\times X} (X\times X)^\sim_i \xleftarrow{\Delta^{\log}_X} X$ ← log diagonal, which is defined by the universal properties of blow up.

$$\begin{array}{ccc} (X\times X)^\sim & \xleftarrow{\Delta^{\log}_X} & X \\ \downarrow & \swarrow_{\Delta_X} & \\ X\times X & & \end{array}$$

Examples 2.2 1) $X = \mathbb{A}^1 \hookleftarrow D = (\text{origin})$



In fact $(X\times X)^\sim = \mathrm{Spec}\, \dfrac{k[T_1, T_2, U^{\pm 1}]}{(T_1 - UT_2)} = X \times \mathbb{G}_m = \mathbb{G}_{m,X}$

The log diagonal $X \xrightarrow{\Delta^{\log}_X} \mathbb{G}_{m,X}$ is the unit section.

2) (X, D) log smooth, WMA $X = \mathbb{A}^d$, $U = \mathbb{G}_m^d$, $D = X \setminus U$.

$$\mathbb{A}^d \times \mathbb{A}^d \xrightarrow{\cong} (\mathbb{A}^2)^d$$
$$((x_i), (y_i)) \longmapsto ((x_i, y_i))$$

<u>Lemma 2.3</u> $X = \operatorname{Spec} A$ with D_i defined by $t_i \in A$

1) $$(X \times X)' = \bigcup_{I = I_1 \sqcup I_2} \operatorname{Spec} \frac{A \otimes_F A[U_i, V_j \,;\, i \in I_1, j \in I_2]}{\left(t_i \otimes 1 - U_i(1 \otimes t_i),\ V_j(t_j \otimes 1) - 1 \otimes t_j\right)_{i \in I_1, j \in I_2}}$$

2) $$(X \times X)^{\sim} = \operatorname{Spec} \frac{A \otimes_F A[U_i^{\pm 1}]_{i \in I}}{(t_i \otimes 1 - (1 \otimes t_i) U_i)_{i \in I}}$$

The log diagonal $X \xrightarrow{\Delta_X^{\log}} (X \times X)^{\sim}$ is given by $a \otimes 1 - 1 \otimes a$, $U_i = 1$ <u>for all</u> $i \in I$ and all $a \in A$.

<u>Remark</u> <u>Why called log?</u>

(X, D) log smooth $\Rightarrow$ $X \to X \times X$, $\Omega_{X/F} = N_{X/X \times X}$

$X \to (X \times X)^{\sim}$, $\Omega_{X/F}(\log D) = N_{X/(X \times X)^{\sim}}$.

$\boxed{\text{2.4 } \underline{\text{Universal Property}} \text{ we only discuss a special case}}$

$(X, D = \bigcup_{i \in I} D_i)$, $(Y, E = \bigcup_{j \in J} E_j)$ log scheme. $X \xrightarrow{f} Y$ such that $f^* E_j = \sum_{i \in I} e_{ij} D_i$ for some $e_{ij} \geq 0$

then $f \times f$ induce a map $(X \times X)^{\sim} \xrightarrow{(f \times f)^{\sim}} (Y \times Y)^{\sim}$

<u>Definition 2.5</u> $(X \times_Y X)^{\sim} = (X \times X)^{\sim} \times_{(Y \times Y)^{\sim}} Y$

<u>Example 2.6</u>

$$\begin{tikzcd} D \arrow[r, hook, "S=0"] & X = \operatorname{Spec} F[S] \arrow[d, "f"] \\ E \arrow[r, hook, "T=0"] & Y = \operatorname{Spec} F[T] \end{tikzcd} \qquad \begin{array}{c} S^n \\ \uparrow \\ T \end{array} \qquad f^* E = n \cdot D$$

defined by $U^n = 1$

$$\begin{tikzcd} (X \times_Y X)^{\sim} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (X \times X)^{\sim} = \operatorname{Spec} k[S, U^{\pm 1}] \arrow[d] \\ Y \arrow[r, hook, "V=1"] & (Y \times Y)^{\sim} = \operatorname{Spec} k[T, V^{\pm 1}] \end{tikzcd} \qquad \begin{array}{cc} S^n & U^n \\ \uparrow & \uparrow \\ T & V \end{array}$$

$$\Rightarrow (X \times_Y X)^{\sim} = \operatorname{Spec} \frac{k[S, U^{\pm 1}]}{(U^n - 1)} = X \times \mu_n$$

$\boxed{\text{2.5 Structure of exceptional divisor}}$ $\mathbb{P}^1$-bundle $\rightsquigarrow$ $\mathbb{G}_m$-bundle as <u>example 2.2</u> shows.

<u>Notation</u> (X, D) log smooth $D = \bigcup_{i \in I} D_i$ $\rightsquigarrow$ $(D_i, \bigcup_{j \neq i} (D_i \cap D_j))$ is also log smooth

$$\begin{tikzcd} E_i^{\circ} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (X \times X)^{\sim} \arrow[d] & & \\ E_i \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (X \times X)' \arrow[d, "p"] & D^{(1)'} \arrow[l, hook'] \arrow[d] & D^{(2)'} \arrow[d] \\ D_i \times D_i \arrow[r, hook] & X \times X & D^{(1)} \arrow[l, hook'] \arrow[d, equal] & D^{(2)} \arrow[d, equal] \\ & & D \times X & X \times D \end{tikzcd}$$

<u>exceptional divisor</u>

<u>proper transform</u>

Put $E = \bigcup_{i \in I} E_i$

注意这个
不是$\mathbb{P}^1$-bundle
而是<u>$\mathbb{P}^1$-bundle</u>的 <u>fiber product</u>

Lemma 2.6. (1) $(X\times X)'$ is a smooth with a SNC divisor $D^{(1)'}\cup D^{(2)'}\cup E$.

$$(X\times X)^\sim = (X\times X)'\setminus D^{(1)'}\cup D^{(2)'} \text{ is also smooth}$$

(2) The projection $E_i\to D_i\times D_i$ induces maps $E_i\longrightarrow (D_i\times D_i)'$ and we have an isomorphism

$$(**)\qquad E_i \xrightarrow{\cong} \mathbb{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}(D_i\times D_i)'$$

$E_i\cap (X\times X)^\sim = E_i^\circ$ is the complement of the two disjoint sections $(D_i\times D_i)^\sim \longrightarrow \mathbb{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}(D_i\times D_i)^\sim$ defined by

the surjections $N_{D_i\times D_i/X\times X}\twoheadrightarrow N_{D_i\times D_i/D_i\times X}$, $N_{D_i\times D_i/X\times X}\twoheadrightarrow N_{D_i\times D_i/X\times D_i}$

(3) In particular (**) induces an isomorphism $E^\circ_{i,D_i} = E_i^\circ\times_{(D_i\times D_i)^\sim} D_i \xrightarrow{\cong}\mathbb{G}_{m,D_i}$

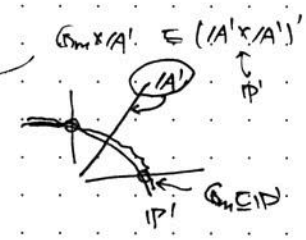
The section $D_i\longrightarrow E^\circ_{i,D_i}$ induced by $X\longrightarrow(X\times X)^\sim$ is identified with the unit section $D_i\longrightarrow\mathbb{G}_{m,D_i}$

(1) is very clear.

Proof (2) reduce to the case $U=\mathbb{G}_m^n \hookrightarrow X=\mathbb{A}^n \hookleftarrow D = X\setminus U$.

$$\begin{array}{ccc} X=\mathbb{A}^n\ (T_1,\dots,T_n) & \hookleftarrow & D=X\setminus U \\ \uparrow{\scriptstyle T_1=0} & & \\ D_1=\mathbb{A}^{n-1} & \hookleftarrow & \bigcup_{j=2}^n (D_1\cap D_j) = Z \end{array}$$

$$\begin{array}{ccc} (X\times X)' = [(\mathbb{A}^1\times\mathbb{A}^1)']^n & \longleftarrow & E_1 = \mathbb{P}^1\times[(\mathbb{A}^1\times\mathbb{A}^1)']^{n-1} = \mathbb{P}^1\times\underline{(D_1\times D_1)'} \\ \downarrow & \square & \downarrow{\scriptstyle \text{by example 2.2}} \\ X\times X = (\mathbb{A}^1\times\mathbb{A}^1)^n & \longleftarrow & D_1\times D_1 = \text{origin}\times(\mathbb{A}^1\times\mathbb{A}^1)^{n-1} \end{array}$$



$$E_1^\circ = E_1\cap(X\times X)^\sim = (\mathbb{P}^1\times[(\mathbb{A}^1\times\mathbb{A}^1)']^{n-1})\cap((\mathbb{G}_m\times\mathbb{A}^1)\times(\mathbb{G}_m\times\mathbb{A}^1)^{n-1})$$
$$= \mathbb{G}_m\times(D_1\times D_1)^\sim$$

∎

§3 Intersection Product with log diagonal, using alteration.

Goal $V\xrightarrow{f}U$ finite étale morphism between smooth schemes over a perfect field F

We want to construct map

$$(\ ,\Delta_{\bar V})^{\log} : CH_d(V\times_U V\setminus\Delta_V)\longrightarrow CH_0(\bar V\setminus V)\otimes\mathbb{Q}.$$

$$\begin{array}{ccc} W & \hookrightarrow & Z \\ \downarrow & \square\ \ g & \downarrow \swarrow \text{alteration} \\ V & \hookrightarrow & Y \ \nwarrow \text{smooth compactification} \\ \downarrow & \square\ \ f & \downarrow \end{array}$$

For $\Gamma\in CH_d(V\times_U V\setminus\Delta_V)$

choose $\Gamma'\in Z_d(\overline{W\times_U W\setminus W\times_V W})$ in $(Z\times Z)^\sim$ such that $\Gamma'|_{W\times_U W\setminus W\times_V W} = (g\times g)^!\Gamma$

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Key

§3.1 ~~First~~ Case.

We consider a Cartesian diagram of separated schemes of finite type $/F$. such that

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "h"] & Z \arrow[d, "\bar{h}"] & D=\bigcup_{i\in I} D_i \arrow[l, hook'] \\ U \arrow[r, hook] & X & B \arrow[l, hook'] \end{tikzcd}$$

[illegible] étale ← h

1) U is the complement of a Cartier divisor $B \subseteq X$

2) ~~B~~ (Z,D) log smooth.

Let T be an open neigh. of Δ_W in $W\times_U W$

We define a map $(\ ,\Delta_Z)_{(Z\times Z)^\sim} : CH_d(\overline{W\times_U W\setminus T}) \longrightarrow CH_0(Z\setminus W)$ as follows:

closure in $(Z\times Z)^\sim$.

Since $\overline{W\times_U W\setminus T} \cap (W\times W) = W\times_U W\setminus T \Rightarrow \overline{W\times_U W\setminus T} \cap \Delta_Z \subseteq Z\setminus W \Rightarrow$ Thus $(\ ,\Delta_Z)_{(Z\times Z)^\sim}$ is well defined.

Proposition 3.2

1) There is a unique factorization

$$\begin{tikzcd} CH_d(\overline{W\times_U W\setminus T}) \arrow[r, "{(\ ,\Delta_Z)_{(Z\times Z)^\sim}}"] \arrow[d] & CH_0(Z\setminus W) \\ CH_d(W\times_U W\setminus T) \arrow[ur, "{\exists!\,(-,\Delta_Z)^{\log}}"'] \arrow[d] & \\ 0 & \end{tikzcd}$$

2) Compactible with Gysin pull-back

For any
$$\begin{tikzcd} W' \arrow[r, hook] \arrow[d, "k"'] \arrow[dr, phantom, "\square"] & Z' \arrow[d, "\bar{k}"] & D' \arrow[l, hook'] \\ W \arrow[r, hook] & Z & \end{tikzcd}$$
(Z',D') log smooth

We have a commutative diagram

$$\begin{tikzcd} CH_d(W\times_U W\setminus T) \arrow[r, "{(-,\Delta_Z)^{\log}}"] \arrow[d] & CH_0(Z\setminus W) \arrow[d, "\bar{k}^!"] \\ CH_{d'}(W'\times_U W'\setminus (k\times k)^{-1}(T)) \arrow[r, "{(-,\Delta_{Z'})^{\log}}"'] & CH_0(Z'\setminus W') \end{tikzcd}$$

proof 2) follows from 1) and the following diagram

$$\begin{tikzcd} (Z'\times Z')^\sim \arrow[d, "(\bar{k}\times\bar{k})^\sim"'] & (Z'\times_X Z')^\sim \arrow[l, hook'] \arrow[d, "(\bar{k}\times\bar{k})^\sim"] \arrow[dl, phantom, "\square"] & Z' \arrow[l, hook'] \arrow[d, "\bar{k}"] \\ (Z\times Z)^\sim & (Z\times_X Z)^\sim \arrow[l, hook'] & Z \arrow[l, hook'] \end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd} CH_d((Z'\times_X Z')^\sim\setminus(\bar{k}\times\bar{k})^{-1}(T)) \arrow[r, "\Delta_{Z'}"] & CH_0(Z'\setminus W') \\ CH_d((Z\times_X Z)^\sim\setminus T) \arrow[u, "(\bar{k}\times\bar{k})^{\sim!}"] \arrow[r, "\Delta_Z"] & CH_0(Z\setminus W) \arrow[u, "\bar{k}^!"'] \end{tikzcd}$$

proof 1)

~~$CH_d(\overline{W\times_U W\setminus T}) \longrightarrow$~~

$$\begin{tikzcd} CH_d(\overline{W\times_U W}\setminus W\times_U W) \arrow[r] \arrow[d] & CH_d(\overline{W\times_U W\setminus T}) \arrow[r, "res"] \arrow[d] & CH_d(W\times_U W\setminus T) \arrow[r] \arrow[d] & 0 \\ CH_d((Z\times_X Z)^\sim\setminus W\times_U W) \arrow[r] \arrow[drr] & CH_d((Z\times_X Z)^\sim\setminus T) \arrow[r] \arrow[d, "{(\ ,\Delta_Z)_{(Z\times Z)^\sim}}"] & CH_d(W\times_U W) \arrow[r] & 0 \\ & CH_0(Z\setminus W) & & \end{tikzcd}$$

we show this a zero map.

But $(Z\times Z)^\sim \setminus W\times W = \bigcup_i E_i^\circ$

But $(Z\times_X Z)^\sim \setminus W\times_U W$

$= \bigcup_i \left((Z\times_X Z)^\sim \cap E_i^\circ\right)$

The pullback of the divisor E_i° by $Z \xrightarrow{\Delta_Z} (Z\times Z)^\sim$ is the Cartier divisor $D_i \subseteq Z$

unit section

$\Delta_Z \ldots D_i \longrightarrow E_i^\circ$

$$\begin{tikzcd}
(Z\times_X Z)^{\sim}\cap E^{\circ}_{i,D_i} \subseteq (Z\times_X Z)^{\sim} \arrow[d] \arrow[r] & (Z\times Z)^{\sim} \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\square"] & (X\times X)^{\sim}
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
D_i \arrow[r, "\text{unit}"] \arrow[dr, equal] & E^{\circ}_{i,D_i} \arrow[r] \arrow[d] & E^{\circ}_i \arrow[d] \\
& D_i \arrow[r] & (D_i\times D_i)^{\sim}
\end{tikzcd}$$

($E^{\circ}_{i,D_i} \cong \mathbb{G}_{m,D_i}$)

$\Rightarrow$

$$\begin{tikzcd}
CH_d((Z\times_X Z)^{\sim}\cap E^{\circ}_i) \arrow[r, "(-,\Delta_{D_i})_{E^{\circ}_i}"] \arrow[d, "(-,\Delta_{D_i})_{(D_i\times D_i)^{\sim}}"] & CH_0(D_i) \\
CH_1((Z\times_X Z)^{\sim}\cap E^{\circ}_{i,D_i}) \arrow[ur, "(\cdot,D_i)_{E^{\circ}_{i,D_i}}"] \arrow[d] & \\
CH_1(\mu_{e,D_i}) \arrow[r] \arrow[uur, bend right, "\Phi"'] & CH_1(\mathbb{G}_{m,D_i}) \arrow[uu, "(\cdot,D_i)_{\mathbb{G}_{m,D_i}}"']
\end{tikzcd}$$

same idea 参 Kato Saito 1.1.b, 40頁

But by example 2.6 $(Z\times_X Z)^{\sim}\cap E^{\circ}_{i,D_i}$ is a closed subscheme of $\mu_{e,D_i} \subseteq E^{\circ}_{i,D_i} = \mathbb{G}_{m,D_i}$ for some integer $e\geq 0$.

Hence, we have a decomposition

But Φ is a zero map $\Rightarrow$ ☒

$\mu_{e,D_i} = \sqcup D_i$

$$\begin{tikzcd}
CH_1(D_i) \arrow[r] \arrow[d, bend right, "0\text{-map}"'] & CH_1(\mathbb{A}^1_{D_i}) \arrow[r] \arrow[d, "\cong", "\text{zero section}"'] & CH_1(\mathbb{G}_{m,D_i}) \arrow[r] \arrow[d, "(\cdot,D_i)_{\mathbb{G}_{m,D_i}}"] & 0 \\
& CH_0(D_i) & CH_0(D_i) &
\end{tikzcd}$$

unit section.

3.2 General Case

$V \xrightarrow[\text{finite étale}]{f} U$ /smooth schemes of dimension d, we don't have
$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$, but we can use an alteration $\rightsquigarrow \cdots$

An alteration $Z\longrightarrow Y$ is a proper, surjective and generically finite map

Theorem 3.3 [de Jong] X/k variety, $Z \subsetneq X$ "proper" subscheme ($Z\neq X$)

Then there is an alteration $Y \xrightarrow{\varphi} X$ ~~such that~~ and an open immersion $Y \xrightarrow{j} \overline{Y}$ such that

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, hook, "j"] & X \\
\overline{Y} & Z \arrow[u, hook]
\end{tikzcd}$$

(1) $\overline{Y}$ projective and regular (2) $j(\varphi^{-1}(Z)) \cup (\overline{Y}\setminus j(Y))$ is a strict normal crossing divisor.

If k is perfect, φ may be chosen to be generically étale.

Lemma 3.4 F: perfect field. Let Y/F be a separated scheme such that $V \underset{\text{dense}}{\overset{\text{open}}{\hookrightarrow}} Y$.

Then there is a commutative diagram such that

(✱✱ alteration)

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\overline{g}"] \arrow[ddr, "h"] & \\
V \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y & \\
U \arrow[rr, hook] & & X
\end{tikzcd}$$

1) $\overline{g}$ is an alteration, g is generically finite, surj, of constant degree $[W:Y]$

2) $B = X\setminus U$ is a Cartier divisor

3) $(Z, D = Z\setminus W)$ log smooth

4) The two ~~quadratic~~ quadrangles are Cartesian.

Proof. F perfect, Nagata's thm $\Rightarrow$ $\exists X/F$ proper, $U \underset{\text{dense}}{\overset{\text{open}}{\subseteq}} X$, blowup at $X\setminus U$, WMA $B = X\setminus U$ is a Cartier divisor.

構造 W, Z

$\rightsquigarrow$ Now replace Y by the closure of the graph $f \subseteq Y\times X$. We may assume there is a commutative diagram $\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$

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Definition 3.5

V/k smooth of finite type ; $\mathcal{C}_V$ = cat. of proper compactifications of V with morphisms which are the identity on V.

$$CH_0(\overline{V}\backslash V) \overset{\text{def}}{=} \varprojlim_{Y\in\mathcal{C}_V} CH_0(Y\backslash V)$$ with proper push forward as transition map.

$$\begin{tikzcd} & CH_0(\overline{V}\backslash V) \arrow[dl] \arrow[dr, "\deg"] & \\ CH_0(Y\backslash V) & & \mathbb{Z} \end{tikzcd}$$

Theorem 3.6 There is a unique map $(\,, \Delta_{\overline{V}})^{\log}: CH_d(V\times_U V\backslash \Delta_V) \longrightarrow CH_0(\overline{V}\backslash V)\otimes\mathbb{Q}$ such that for any (** alteration) and any $Y\in\mathcal{C}_V$, we have

$$\begin{tikzcd} CH_d(V\times_U V\backslash\Delta_V) \arrow[r, "(\,,\Delta_{\overline{V}})^{\log}"] \arrow[d, "(g\times g)^!"] & CH_0(\overline{V}\backslash V)\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[r] & CH_0(Y\backslash V)\otimes_{\mathbb{Z}}\mathbb{Q} \\ CH_d(W\times_U W\backslash W\times_V W) \arrow[r, "(\,,\Delta_Z)^{\log}"', "\text{prop 3.2}"] & CH_0(Z\backslash W) \arrow[ur, "\frac{1}{[W:V]}\overline{g}_*"'] & \end{tikzcd}$$

proof Only need to note that $W\times_V W$ is an open neigh. of Δ_W in $W\times_U W$.

$$\begin{tikzcd} \Delta_W \arrow[r] & W\times_V W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W\times_U W \arrow[d] & \\ & V \arrow[r, "\text{open and closed}"'] \arrow[dr, bend right, equal] & V\times_U V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & & V \arrow[r, "\text{finite étale}"'] & U \end{tikzcd}$$

Example 3.7

$$\begin{tikzcd} & V \arrow[r, "f", "\text{finite étale}"'] \arrow[d, hook] & U \arrow[d, hook] & \\ \text{normalization of } X \text{ in } V \arrow[r] & Y \arrow[r] & X & \text{smooth compactification} \\ & Y\backslash V = D \arrow[u, hook] & B = X\backslash U \arrow[u, hook] & \end{tikzcd}$$

/ smooth connected curves.

σ : non-trivial automorphism of V over U which extends to an auto. of Y over X

$\Gamma_\sigma \subseteq V\times_U V$ be the graph of σ.

$\boxed{\Gamma_\sigma \subseteq V\times_U V\backslash V \text{ assume } V^\sigma = \text{empty.}}$

$$Y^\sigma_{\log} = \Delta_Y \cap \Big(Y\backslash \bigcup_{\substack{\sigma(y)\neq y\\ y\in D}} \{y\} \xrightarrow{\tilde{\Gamma}_\sigma} (Y\times Y)^\sim\Big)$$

log σ-fixed part

which is an extension of Γ_σ to $(Y\times Y)^\sim$

then

$$(\Gamma_\sigma, \Delta_{\overline{V}})^{\log} = \sum_{\substack{y\in D\\ \sigma(y)=y}} \text{length}\ \frac{\widehat{\mathcal{O}}_{Y,y}}{\widehat{\mathcal{O}}_{Y,y}\left(\frac{\sigma(b)}{b}-1\,;\, b\in\widehat{\mathcal{O}}_{Y,y}\backslash\{0\}\right)}\cdot[y]$$

proof

$$\begin{tikzcd} Y^\sigma_{\log} \arrow[r, "\mathcal{I}"] & (Y\times Y)^\sim \\ & Y \arrow[u, "\Delta_Y"] \end{tikzcd}$$

Now $(y,y)\in Y^\sigma_{\log}\cap\Delta_Y$ if $\sigma(y)=y$, then

and the ideal $\mathcal{I}$ at (y,y) is generated by $a\otimes 1 - 1\otimes\sigma(a)$ for all a, $b\neq 0$

$\frac{\sigma(a\otimes 1)}{a\otimes 1}-1$, $\frac{\sigma(1\otimes b)}{1\otimes b}$ - -

$\frac{\sigma(b)}{b}-1$

Hence $(\Gamma_\sigma, \Delta_{\overline{V}})^{\log}_{\substack{y\\ \uparrow\\ \text{at } y}} \underset{\text{Fulton}}{=} \text{length}\ \frac{\widehat{\mathcal{O}}_{Y,y}}{\widehat{\mathcal{O}}_{Y,y}\left(\frac{\sigma(b)}{b}-1\,,\, b\in\widehat{\mathcal{O}}_{Y,y}\backslash\{0\}\right)}$

§4. Swan classes of étale sheaves

Notation Let $V \xrightarrow{f} U$ be a finite étale covering of Galois group G between smooth schemes, purely of dimension d.

We have $V \times_U V = \coprod_{\sigma \in G} \Gamma_\sigma$, $CH_d(V \times_U V \setminus \Delta_V) \cong$ generated by $G \setminus \{1\}$. (identify with the free abelian group)

Γ_σ = graph of $\sigma \subseteq V \times_U V$.

$$Sw(f_* \mathbb{F}_\ell)$$
$$\| $$
$$\boxed{\underline{\text{Wild discriminant}} \; \boxed{d^{\log}_{V/U} = f_* D^{\log}_{V/U}}} \in CH_0(\bar{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Definition 4.1 [Swan character class]

Wild different of V/U, generalizing the classical case for curves.

$$G \xrightarrow{S_{V/U}} CH_0(\bar{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\sigma \longmapsto S_{V/U}(\sigma) = \begin{cases} D^{\log}_{V/U} = (V \times_U V \setminus \Delta_V, \Delta_V)^{\log} & \sigma = 1 \\ -(\Gamma_\sigma, \Delta_{\bar{V}})^{\log} & \sigma \neq 1 \end{cases}$$

By definition $\sum_{\sigma \in G} S_{V/U}(\sigma) = 0$ and if the order of σ is not a power of the char p of F, we have $S_{V/U}(\sigma) = -(\Gamma_\sigma, \Delta_{\bar{V}})^{\log} = 0$

Theorem 4.2 [Trace formula]

$$\deg S_{V/U}(\sigma) = \begin{cases} [V:U]\, \chi_c(U_{\bar{F}}, \mathbb{Q}_\ell) - \chi_c(V_{\bar{F}}, \mathbb{Q}_\ell) & \text{if } \sigma = 1 \\ -\mathrm{Tr}\left(\sigma^* : H^*_c(V_{\bar{F}}, \mathbb{Q}_\ell)\right) & \text{if } \sigma \neq 1 \end{cases}$$

The first identity is a generalization of Hurwitz theorem for curves.

[If X, Y proper smooth curve, $X \to Y$ finite separable, then $2g(X) - 2 = [X:Y](2g(Y) - 2) + \deg R$

$$R = \sum_{p \in X \setminus V} \text{length}(\Omega_{X/Y})_p \cdot [p] \quad \text{supported on ramification locus}$$

$$\text{and } D^{\log}_{V/U} = \sum_{y \in X \setminus V} \left[\text{length}(\Omega_{X/Y})_p - (e_{X/Y,x} - 1) \right]$$

ramification index at x

$\text{length}_{\hat{\mathcal{O}}_{X,x}} \Omega^1_{\hat{\mathcal{O}}_{X,x}/\hat{\mathcal{O}}_{Y,f(x)}}$.]

Definition 4.3 $\mathcal{F}$: smooth $\mathbb{F}_\ell$-sheaf on U such that $\mathcal{F}|_V$ is the constant sheaf M.

$f_* : CH_0(\bar{V} \setminus V) \to CH_0(\bar{U} \setminus U)$

"naive" Swan class of $\mathcal{F}$
$$Sw'(\mathcal{F}) = \frac{1}{|G|} f_* \left(\sum_{\sigma \in G_{(p)}} S_{V/U}(\sigma) \cdot \mathrm{Tr}^{Br}(\sigma : M) \right) \in CH_0(\bar{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$$

order = power of p

$$Sw(\mathcal{F}) = \frac{1}{|G|} f_* \left(\sum_{\sigma \in G_{(p)}} \left(\dim_{\mathbb{F}_\ell} M^\sigma - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma^p}/M^\sigma}{p-1} \right) \cdot S_{V/U}(\sigma) \right)$$

$$\in CH_0(\bar{U} \setminus U) \otimes \mathbb{Q}.$$

Thm 4.4 (GOS formula) $\chi_c(U_{\bar{F}}, \mathcal{F}) = \operatorname{rank}\mathcal{F} \cdot \chi_c(U_{\bar{F}}) - \deg \operatorname{Sw}'(\mathcal{F})$

proof Brauer thm, $\chi_c(U_{\bar{F}}, \mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G_{(p)}} \operatorname{Tr}\left(\sigma^* : H_c^*(V_{\bar{F}}, \mathbb{Q}_\ell)\right) \cdot \operatorname{Tr}^{Br}(\sigma : M))$ Thm 4.2 $\Rightarrow$ Thm 4.4 ⊠

For an $\bar{\mathbb{F}}_\ell$-automorphism σ of an $\bar{\mathbb{F}}_\ell$-vector space M of dimension m, the Brauer trace $\operatorname{Tr}^{Br}(\sigma : M) \in \mathbb{Z}[\zeta_\infty] \subseteq \bar{\mathbb{Q}}_\ell$ is defined as follows: Let $\alpha_1, \dots, \alpha_m$ be the eigenvalues of σ counted with multiplicities and

? let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_m \in \mathbb{Z}[\zeta_\infty] \subseteq \bar{\mathbb{Q}}_\ell$ be the roots of unity of order prime to ℓ lifting $\alpha_1, \dots, \alpha_m$

Then we define $\operatorname{Tr}^{Br}(\sigma : M) = \sum_{i=1}^{m} \tilde{\alpha}_i$

Remark If σ is an auto. of order p^e of M, then $|(\mathbb{Z}/p^e\mathbb{Z})^\times| \cdot \left(\dim_{\bar{\mathbb{F}}_\ell} M^\sigma - \frac{\dim_{\bar{\mathbb{F}}_\ell} M^{\sigma^p}/M^p}{p-1}\right) = \sum_{i \in (\mathbb{Z}/p^e\mathbb{Z})^\times} \operatorname{Tr}^{Br}(\sigma^i : M)$

Example 4.5 Wild differential and Wild discriminant.

§5 Lefschetz trace formula ~~and~~ for open varieties

Theorem 5.1 $X/_F$ proper smooth, purely of dimension d. $U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X \overset{\text{SNC}}{\hookleftarrow} D = X \setminus U$.

$(X \times X)' \xrightarrow{p} X \times X$ $D^{(1)\prime}, D^{(2)\prime}$ proper transform of $D^{(1)} = D \times X$, $D^{(2)} = X \times D$.

$\Gamma \subseteq U \times U$ closed, purely of dimension d.

$\bar{\Gamma}'$: closure of Γ in $(X \times X)'$

Assume that $\bar{\Gamma}' \cap D^{(1)\prime} \subseteq \bar{\Gamma}' \cap D^{(2)\prime}$ (**)

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "p_1"'] \arrow[d, hook] \arrow[dr, "p_2"] \arrow[ddl, bend right=60, "g"'] & \\
U \arrow[dr, "f"'] & U \times U \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & U \arrow[dl, "f"] \\
 & \operatorname{Spec} F &
\end{tikzcd}$$

Then ① $p_2 : \Gamma \to U$ is proper ($\Rightarrow \Gamma^* = p_{1*} \circ p_2^*$ well defined)

② $\operatorname{Tr}(\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell)) = \deg(\bar{\Gamma}', \Delta'_X)_{(X \times X)'}$

(**) $\Rightarrow$ ① We use the following Lemma

Lemma 5.2 p_2 is proper iff $\bar{\Gamma} \cap D^{(1)} \subseteq \bar{\Gamma} \cap D^{(2)}$. where $\bar{\Gamma}$ is the closure of Γ in $X \times X$.

The above Lemma follows from Valuation Criterion ~~of~~ for properness. $(X \times X)' \setminus U \times U = D^{(1)\prime} \cup D^{(2)\prime} \cup E$

Now, since $\bar{\Gamma}' \cap (D^{(1)\prime} \cup E) \subseteq \bar{\Gamma}' \cap (D^{(2)\prime} \cup E) \Rightarrow \bar{\Gamma}' \setminus (U \times U) \cap \bar{\Gamma}' \subseteq \bar{\Gamma}' (\bar{D}^{(2)\prime} \cup E)$

take image by $p \Rightarrow \underset{\overset{\|}{\bar{\Gamma} \setminus \Gamma}}{\bar{\Gamma} \setminus (U \times U) \cap \bar{\Gamma}} \subseteq \bar{\Gamma} \cap D^{(2)} \Rightarrow \bar{\Gamma} \cap D^{(1)} \subseteq \bar{\Gamma} \cap D^{(2)}$.

5.3 Definition of $Tr(\Gamma^*: H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell))$ if P_2 is proper

Since P_2 is proper, the pull-back ~~is defined~~ $P_2^*: H_c^q(U_{\bar{F}}, \mathbb{Z}_\ell) \longrightarrow H_c^q(\Gamma_{\bar{F}}, \mathbb{Z}_\ell)$ is defined by $Rf_! \mathbb{Z}_\ell \to Rf_! RP_{2*} P_2^* \mathbb{Z}_\ell = Rg_! \mathbb{Z}_\ell$.

Now we define push forward $P_{1*}: H_c^q(\Gamma_{\bar{F}}, \mathbb{Z}_\ell) \longrightarrow H_c^q(U_{\bar{F}}, \mathbb{Z}_\ell)$

The trace map $Tr_g: Rg_! \mathbb{Z}_\ell(d)[2d] \longrightarrow \mathbb{Z}_\ell$ induce $\mathbb{Z}_\ell(d)[2d] \longrightarrow Rg^! \mathbb{Z}_\ell = RP_1^! Rf^! \mathbb{Z}_\ell \overset{f:\text{smooth}}{=\!=} RP_1^! \mathbb{Z}_\ell(d)[2d]$

$\Rightarrow \mathbb{Z}_\ell \longrightarrow RP_1^! \mathbb{Z}_\ell \rightsquigarrow RP_{1!} \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell \rightsquigarrow Rg_! \mathbb{Z}_\ell = Rf_! RP_{1!} \mathbb{Z}_\ell \longrightarrow Rf_! \mathbb{Z}_\ell \Rightarrow P_{1*}: H_c^q(\Gamma_{\bar{F}}, \mathbb{Z}_\ell) \to H_c^q(U_{\bar{F}}, \mathbb{Z}_\ell)$.

Now put $\Gamma^* = P_{1*} \circ P_2^*$

5.4 idea of Proof of ②

$j: U \to X$

We first have a look of the classical case

If $X = U$ is proper

$$\begin{tikzcd}
\Gamma^* \in \bigoplus_q \mathrm{End}\, H^q(X, \mathbb{Z}_\ell) \arrow[d, "Tr"] & H^{2d}(X \times X, \mathbb{Z}_\ell(d)) \arrow[l, "\text{Poincaré dual}"', "\text{Künneth}"] \arrow[d, "\Delta^*"] \\
Tr(\Gamma^*) \in \mathbb{Q}_\ell & H^{2d}(X, \mathbb{Q}_\ell(d)) \arrow[l, "Tr"]
\end{tikzcd}$$

$\underline{\underline{cl(\Gamma)}} \in H^{2d}(X \times X, \mathbb{Z}_\ell(d))$

$Tr(\Gamma^*) = Tr(\Delta^*(cl(\Gamma)))$.

However, cycle class map is compactible with the intersection product

$\Rightarrow \Delta^*(cl(\Gamma)) = cl(\Gamma, \Delta)_{X \times X}$

$\Rightarrow Tr(\Gamma^*) = \overset{Tr}{\cancel{\deg}}\, cl(\Gamma, \Delta)_{X \times X} \overset{\text{def}}{=} (\Gamma, \Delta)_{X \times X}$

for open variety case

We define $H^p_{!*}(U \times U, \mathbb{Z}_\ell(d)) = H^p(X \times U, (j \times id)_! \mathbb{Z}_\ell(d))$

$$\begin{tikzcd}
\Gamma^* \in \bigoplus \mathrm{End}(H_c^q(U_{\bar{F}}, \mathbb{Z}_\ell)) \arrow[d, "Tr"] & H^{2d}_{!*}(U \times U, \mathbb{Z}_\ell(d)) \arrow[l, "\text{Poincaré dual}"', "\text{Künneth}"] \arrow[d, "\Delta^*"] \\
Tr(\Gamma^*) \in \mathbb{Q}_\ell & H_c^{2d}(U_{\bar{F}}, \mathbb{Z}_\ell(d)) \arrow[l, "Tr"]
\end{tikzcd}$$

Key points

① $[\Gamma] \in H^{2d}_{!*}(U \times U, \mathbb{Z}_\ell(d))$ well defined. (if $P_2: \Gamma \to U$ proper)

$\Rightarrow Tr(\Gamma^*) = Tr(\Delta^*[\Gamma^*])$

②

② image of $[\Gamma]$ is Γ^*

③ $\Delta^*([\Gamma]) = cl(\bar{\Gamma}', \Delta'_X)_{(X \times X)'}$ ← classical case.

④ $[\Gamma] \overset{\text{map to}}{\rightsquigarrow} [\Gamma']$ as below.

$$\begin{tikzcd}
\Delta^*([\Gamma]) \rightsquigarrow & {[\Gamma] \in H^{2d}_{!*}(U_{\bar{F}} \times U_{\bar{F}}, \mathbb{Z}_\ell(d))} \arrow[r, "\Delta^*"] \arrow[d, "\text{1988 Falting's Lemma}"] & H_c^{2d}(U_{\bar{F}}, \mathbb{Z}_\ell(d)) \arrow[d] \\
& {[\Gamma'] \in H^{2d}((X \times X)' \setminus D^{(2)'}, \mathbb{Z}_\ell(d))} \arrow[r, "\Delta'^*"] & H^{2d}(X_{\bar{F}}, \mathbb{Z}_\ell(d)) \arrow[u, equal]
\end{tikzcd}$$

where $\Gamma' = \bar{\Gamma}' \setminus \bar{\Gamma}' \cap D^{(2)'}$

Since $P_2: \Gamma \to U$ is proper

$\Rightarrow \Gamma \underset{\text{closed}}{\subseteq} X \times U$

$$\begin{tikzcd}
\Gamma \arrow[d, "\text{proper}"] \arrow[r, phantom, "\subseteq"] & X \times U \arrow[dl, "\text{proper}"] \\
U &
\end{tikzcd}$$

$\Rightarrow H_\Gamma^{2d}(X \times U, (j \times id)_! \mathbb{Z}_\ell(d))$

$= H_\Gamma^{2d}(X \times U, \mathbb{Z}_\ell(d)) = H_\Gamma^{2d}(U \times U, \mathbb{Z}_\ell(d))$

$\Rightarrow$ Thus the cycle class

$[\Gamma] \in H_\Gamma^{2d}(U \times U, \mathbb{Z}_\ell(d))$

defines a class

$[\Gamma] \in H^{2d}(X \times U, (j \times id)_! \mathbb{Z}_\ell(d))$